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A NEW VARIANT OF EDGE STABILITY IN GRAPHS

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Abstract

In a graph, a vertex may m-dominate an edge and an edge may m-dominate a vertex. In this paper, we have defined the m-adjacency of two vertices by using the concept of an edge m-dominating a vertex. We also define m-adjacent edges using this concept. Further we introduce the concepts of m-edge stable sets, m-edge independent sets and m-edge dominating sets. These concepts are variants of known concepts of edge stable sets, edge independent sets and edge dominating sets in graphs. The complement of the every m-edge stable set is an ev-dominating set. We prove characterizations of maximal m-edge stable sets and maximal m$-$edge independent sets. We deduce that every maximal m-edge stable set is an m-edge dominating set. We also provide a characterization of minimal m-edge dominating set.

1. Introduction

In the area of mixed domination, several new concepts have appeared in the literature.

Key Words : *m-edge stable set, Maximal m-edge stable set, m-edge dominating set, m-adjacent vertices, m-adjacent edges, m-edge independent set.*

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For example, the concept of a vertex which m-dominates an edge and the concept of an edge which m-dominates a vertex have been defined an studied by some authors like Laskar, R., Peters, K., Sampathkuamr, E., Kamath, S. S., and others [1, 2, 3]. We observe that these concepts give rise to new types of adjacency of vertices and edges in graphs. Further these new concepts can be used to define new domination related parameters like m-edge domination, m-edge stability, m-edge independence and others. In fact one such concept called ev-domination has already been defined in [2]. In this paper, we define these new concepts and relate them with already existing concepts like edge domination, edge stability, edge independence and others [4,5].

2. Preliminaries and Notations

If G is a graph then $E(G)$ denotes the edge set and $V(G)$ denotes the vertex set of the graph. If S is any set then S denotes the cardinality of S. If f is an edge of G then $G \setminus f$ denotes the subgraph of G obtained by removing the edge f. $N(V)$ denotes the set of vertices which are adjacent to v. $N[v] = N(V) \cup \{v\}$. If x is any vertex then $d(x)$ denotes the degree of x and is the number of edges incident at x.

Definition 2.1 : An edge $e = uv$ m-dominates a vertex x if $x \in N(u) \cup N(v)$. [2]

Definition 2.2 : A set $T \subset E(G)$ is an ev-dominating set if every vertex in G is m-dominated by an edge in T. An ev-dominating set of minimum cardinality is called a minimum ev-dominating set. The ev-domination number for the graph G is denoted by $\gamma_{ev}(G)$ and is the cardinality of a minimum ev-dominating set. [2]

Definition 2.3 : An edge $g = uv$ be e-dominates the edge $h = xy$ if $x, y \in N[u] \cup N[v]$. [6]

Definition 2.4 : A set F of edges is said to be an edge-edge dominating (EED) set of G if for every edge h not in F, there is an edge g in F such that h is e-dominated by g. An edge-edge dominating set with minimum cardinality is called minimum edge-edge dominating set. The cardinality of minimum edge-edge dominating set is called edge-edge domination number and it is denoted by $\gamma_{ee}(G)$. [6]

Definition 2.5 : (edge dominating set) Let G be a graph and F be a set of edges of G. Then F is said to be an edge dominating set if for every $e \in E(G) \setminus F$ there is some edge f in F such that e and f are adjacent. [5]

Definition 2.6 : (edge stable set) Let G be a graph and F be a set of edges of G .

Then F is said to be an edge stable set if for every vertex x for which $d(x) \geq 2$, there is an edge e containing x such that $e \notin F$. An edge stable set F is said to be a maximal edge stable set if for every edge in $E(G) \setminus F$, $F \cup \{e\}$ is not an edge stable set. An edge stable set with maximum cardinality is called a maximum edge stable set and its cardinality is called the edge stability number of the graph and it is denoted as $\beta'(G)$. [4]

Now, we introduce the following definitions.

Definition 2.7 : (m-adjacent vertices) Let u and v be two vertices of G. Then u and v are said to be m-adjacent vertices in G if there is an edge of G which m-dominates both u and v in G.

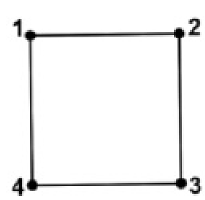
Obviously, adjacent vertices are always m-adjacent vertices but m-adjacent vertices need not be adjacent vertices.

Example : Consider the path graph P_3 with vertices $\{1, 2, 3\}$. Then vertices 1 and 3 are m-adjacent vertices but they are not adjacent vertices.

P_3

Definition 2.8 : (m-adjacent edges) Let e and f be two edges of G. Then e and f are said to be m-adjacent edges in G if they m-dominate some vertex of G. If two edges are adjacent then they are m-adjacent but the converse is not true.

Example : Consider the cycle graph C_4 with vertices $\{1, 2, 3, 4\}$



C_4

Let $e = \{12\}$ and $f = \{34\}$. Then e and f are m-adjacent but they are not adjacent.

Note that if an edge f e-dominates an edge g or g e-dominates f then f and g are m-adjacent edges. However, if two edges f and g are m-adjacent then it may happen that f does not e-dominate g and g does not e-dominate f.

Definition 2.9 : (m-edge degree) Let G be a graph and $v \in V(G)$. Then m-edge degree of v is the number of edges which m-dominate v. It is denoted as $d_{me}(x)$.

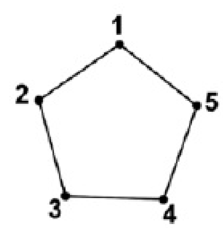
Note that if v is an end vertex of an edge e then e m-dominates v. Therefore, $d(v) \leq d_{me}(v)$.

The maximum m-edge degree of a graph is denoted as $\Delta_{me}(G)$ and the minimum m-edge degree of a graph is denoted as $\delta_{me}(G)$.

Definition 2.10 : (m-edge stable set) Let G be a graph and F be a set of edges of G. Then F is said to be an m-edge stable set if for every vertex x with $d_{me}(x) \geq 2$, there is an edge e which m-dominates x such that $e \notin F$.

Note that every edge stable set is an m-edge stable set. But m-edge stable set need not be an edge stable set.

Example : Consider the cycle graph C_5 with vertices $\{1, 2, 3, 4, 5\}$.



C_5

Let $F = \{12, 23\}$. Then F is an m-edge stable set. Note that F is not an edge stable set.

Note that to be m-edge stable set is hereditary property.

Definition 2.11 : (maximal m-edge stable set) An m-edge stable set F is said to be a maximal m-edge stable set if for every edge e not in F, $F \cup \{e\}$ is not an m-edge stable set.

Definition 2.12 : (maximum m-edge stable set) An m-edge stable set with maximum cardinality is called a maximum m-edge stable set. The cardinality of a maximum m-edge stable set is called the m-edge stability number of the graph and it is denoted as $\beta'_{ms}(G)$.

Example : Let G be a graph with $\delta_{me}(G) = k$ where $k \geq 2$. Let F be any collection of j edges where $j < k$. Let x be a vertex of G. Then $d_{me}(x) \geq k$. Since F contains less than k edges, there is an edge h which m-dominates x and $h \notin F$. This proves that F is an m-edge stable set of G. Therefore, m-edge stability number of G is greater than or equal to $k - 1$.

Definition 2.13 : (m-isolated edge) Let G be a graph and $e = uv$ be an edge of G. Then e is said to be an m-isolated edge of G if $d_{me}(u) = 1$ and $d_{me}(v) = 1$.

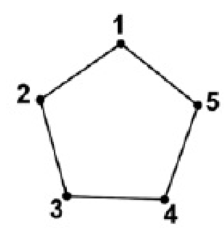
Obviously, an edge e is m-isolated edge if and only if it is an isolated edge.

Definition 2.14 : (m-edge dominating set) Let G be a graph and F be a set of edges of G. Then F is said to be an m-edge dominating set if for every $e \in E(G) \setminus F$ there is some edge f in F such that e and f are m-adjacent.

Note that any superset of m-edge dominating set is an m-edge dominating set but subset of an m-edge dominating set need not be an m-edge dominating set.

Obviously, every edge dominating set is an m-edge dominating set but converse is not true.

Example : Consider the cycle graph C_5 with vertices {1, 2, 3, 4, 5}.



C_5

Let $F = \{12\}$. Then F is an m-edge dominating set but not an edge dominating set.

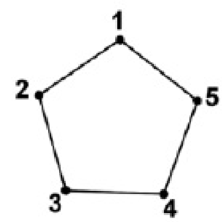
Definition 2.15 : (minimal m-edge dominating set) An m-edge dominating set F is said to be a minimal m-edge dominating set if $F \setminus \{e\}$ is not an m-edge dominating set for every e in F.

Definition 2.16 : (minimum m-edge dominating set) An m-edge dominating set with minimum cardinality is called a minimum m-edge dominating set and it's cardinality is called the m-edge domination number of the graph and it is denoted as $\gamma'_{me}(G)$.

Definition 2.17 : (m-edge independent set) Let G be a graph and F be a set of edges of G. Then F is said to be an m-edge independent set if no two distinct edges of F are

m-adjacent. Obviously, every m-edge independent set is an edge independent set but the converse is not true.

Example : Consider the cycle graph C_5 with vertices {1, 2, 3, 4, 5}.

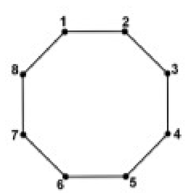


C_5

Let $F = \{12, 34\}$. Then F is not an edge independent set but not m-edge independent set.

Definition 2.18 : (maximal m-edge independent set) An m-edge independent set F is said to be a maximal m-edge independent set if for every edge e not in F, $F \cup \{e\}$ is not an m-edge independent set.

Example : Consider the cycle graph C_8 with vertices {1, 2, 3, 4, 5, 6 ,7 ,8}.



C8

Let $F = \{12, 56\}$. Then F is a maximal m-edge independent set.

Definition 2.19 : (maximum m-edge independent set) An m-edge independent set with maximum cardinality is called maximum m-edge independent set.

3. Main Results

Proposition 3.1 : Let G be a graph and F be a set of edges of G.

(1) If F is an edge dominating set then it is an edge-edge dominating set.

(2) If F is an edge-edge dominating set then it is an m-edge dominating set.

Proof :

(1) Let $e \in E(G) \setminus F$. Since F is an edge dominating set, there is an edge f in F such that e is adjacent to f. Obviously, e is e-dominated by f. Therefore, F is an edge-edge dominating set.

(2) Suppose F is an edge-edge dominating set of G. Let $e = uv \in E(G) \setminus F$. There is an edge $f = xy \in F$ such that e is e-dominated by f. This means that $u \in N(x) \cup N(y)$ and $v \in N(x) \cup N(y)$. Therefore, $f = xy$ and $e = uv$ m-dominate u. Therefore, e and f are m-adjacent edges with $f \in F$ and $e \in E(G) \setminus F$. Thus, F is an m-edge dominating set of G.

Proposition 3.2 : Let G be a graph. If F be any m-edge stable set of G then $E(G) \setminus F$ is an ev-dominating set of G.

Proof : Let x be any vertex of G. Since F is an m-edge stable set of G, there is an edge e which m-dominates x and $e \notin F$. Then $e \in E(G) \setminus F$ and e m-dominates x. Thus, $E(G) \setminus F$ is an ev-dominating set of G.

Theorem 3.3 : Let G be a graph and F be an m-edge stable set of G. Then F is a maximal m-edge stable set of G if and only if following two conditions are satisfied

(1) F contains all m-isolated edges of G.

(2) For every $e \in E(G) \setminus F$, there is a vertex x with $d_{me}(x) \geq 2$ which is m-dominated by e such that all the edges which m-dominate x (except e) are in F.

Proof : Suppose F is a maximal m-edge stable set of G.

(1) Suppose there is an m-isolated edge $e = uv$ of G such that $e \notin F$.

Let $F_1 = F \cup \{e\}$. Let x be any vertex whose m-edge degree is greater than or equal to 2. Then there is an edge f which m-dominates x and $f \notin F$. Now, $f \neq e$ because the $d_{,e}(u) = 1, d_{me}(v) = 1$ and e m-dominates only u and v while f m-dominates x whose m-edge degree is greater than or equal to 2.

Thus, $f \notin F_1$. Thus F_1 is an m-edge stable set of G which contradicts the maximality of F. Thus, F contains all m-isolated edges of G.

(2) Let $e \in E(G) \setminus F$ then e is not an m-isolated edge of G. Since F is a maximal m-edge stable set of G , $F \cup \{e\}$ is not an m-edge stable set of G. Therefore, there is a vertex such that $d_{me}(x) \geq 2$ and all the edges which m-dominate x are in $F \cup \{e\}$. But F is an m-edge stable set of G. Therefore, there is an edge h which m-dominates x and $h \notin F$. This implies that $h = e$. Thus, condition (ii) is satisfied.

Conversely, suppose (i) and (ii) are satisfied.

Let $e \in E(G) \setminus F$. Then e cannot be an m-isolated edge of G. By condition (ii) there is a vertex x whose m-edge degree is greater than or equal to 2 and all the edges which m-dominate x except e are in F. Thus, all the edges which m-dominate x are in $F \cup \{e\}$. Therefore, $F \cup \{e\}$ is not an m-edge stable set. Thus, F is a maximal m-edge stable set.

Corollary 3.4 : Let G be a graph with $\delta(G) \geq 2$. If F is a maximal m-edge stable set of G then it is an ev-dominating set of G.

Proof : Let $x \in V(G)$. Let e be any edge whose end vertex is x. If $e \in F$ then x is m-dominated by some member of F. Suppose $e \notin F$. Then by the above theorem there is a vertex z such that all the edges which m-dominate z (except e) are in F. If $z = y$ or $z = x$ then since $d(x) \geq 2$, there is an edge h whose end vertex is z and $h \neq e$. Also h m-dominates z. Therefore, $h \in F$. Thus x is m-dominated by h and $h \in F$.

Suppose, $z \neq x$ and $z \neq y$.

Since z is m-dominated by e, zx is an edge or zy is an edge. Then $zx \in F$, if zx is an edge and $zy \in F$, if zy is an edge. Thus, x is m-dominated by some member of F. Thus, F is an ev-dominating set of G.

Proposition 3.5 : Let G be a graph and F be a maximal m-edge stable set. Then F is an m-edge dominating set.

Proof : Let $e \in E(G) \setminus F$. Since F is a maximal m-edge stable set, e cannot be an m-isolated edge. By theorem 3.3, there is a vertex x such that $d_{me} \geq 2$ and all the edges which m-dominate x are in F. Since $d_{me}(x) \geq 2$, there is an edge f such that $f \neq e$ and f m-dominates x. Then $f \in F$. Since f and e both m-dominate x, e and f are m-adjacent and $f \in F$. Thus F is an m-edge dominating set of G.

Theorem 3.6 : Let G be a graph and F be an m-edge dominating set of G. Then F is a minimal m-edge dominating set of G if and if only if for every edge $e \in F$ at least one of the following two conditions holds

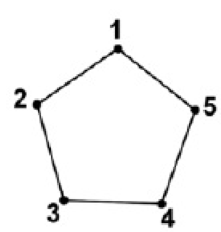
1. e is not m-adjacent to any other edge of F.
2. There is an edge h in $E(G)\setminus F$ such that h is m-adjacent to e but it is not m-adjacent to any other edge of F.

Proof : Suppose F is a minimal m-edge dominating set. Let $e \in F$. Now $F \setminus \{e\}$ is not an m-edge dominating set of G. Therefore, there is an edge f such that $f \notin F \setminus \{e\}$ and f is not m-adjacent to any member of $F \setminus \{e\}$.
Case (i) : If $F = \{e\}$ then the above statement implies that e is not m-adjacent to any other member of F. Thus condition (i) is satisfied.
Case (ii) : If $F \neq \{e\}$ then $f \notin F$. Let $h = f$. Now h is not m-adjacent to any other member of $F \setminus \{e\}$. But F is an m-edge dominating set. Therefore, h is m-adjacent to some member of F. This member must be e. Thus h is m-adjacent to only one member of F namely e. Thus condition (ii) is satisfied.
Conversely, to prove that F is a minimal let $e \in F$.

(a) Suppose condition (i) is true. Then $F \setminus \{e\}$ is not an m-edge dominating set.

(b) Suppose condition (ii) holds. Let $h \in E(G) \setminus F$ such that h is m-adjacent to only one edge namely e. Then $F \setminus \{e\}$ is not an m-edge dominating set of G. Thus F is a minimal m-edge dominating set of G.

Remark 3.7 : Let F be an m-edge independent set. Let x be a vertex of G such that $d_{me}(x) \geq 2$. Let e_1 and e_2 be two edges which m-dominate x. Then e_1 and e_2 are m-adjacent edges. Since F is an m-edge independent set, $e_1 \notin F$ or $e_2 \notin F$. It follows that there is an edge e which m-dominates x and $e \notin F$. Thus, F is an m-edge stable set. Converse is however not true.
Example : Consider the cycle graph C_5 with vertices {1, 2, 3, 4, 5}.



C_5

Let $F = \{12, 23\}$ then F is an m-edge stable set but it is not an m-edge independent set.

Theorem 3.8 : Let G be a graph and F be an m-edge independent set. Then F is a maximal m-edge independent set if and only if F is an m-edge dominating set of G.

Proof : Suppose F is a maximal m-edge independent set. Let $e \in E(G) \setminus F$. Now $F \cup \{e\}$ is not an m-edge independent set. Therefore, there are at least two edges in $F \cup \{e\}$ which are m-adjacent. Since F is an m-edge independent set, one of these edges must be e. Thus, e is m-adjacent with some edge of F. Thus, F is an m-edge dominating set of G.

Conversely, suppose F is an m-edge dominating set. Let $e \in E(G) \setminus F$. Then there is an edge $f \in F$ such that e and f are m-adjacent. Thus, there are edges in $F \cup \{e\}$ which are m-adjacent. Thus, F is a maximal m-edge independent set.

Theorem 3.9 : Let G be a graph and F be an m-edge independent set. Then F is a maximal m-edge independent set if and only if F is a minimal m-edge dominating set.

Proof : Suppose, F is a maximal m-edge independent set. It is already proved that F is an m-edge dominating set. Let $e \in F$. Since F is an m-edge independent set, e is not m-adjacent to any other edge of F. Thus by (i) of theorem 3.6, F is a minimal m-edge dominating set.

Conversely, suppose that F is a minimal m-edge dominating set. Then by the above theorem, F is a maximal m-edge independent set.

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