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FUZZY COMPACT ORDERED SPACES

VINAYAK E. NIKUMBH
Department of Mathematics,
P. V. P. College, Pravaranagar,India

Abstract

In this paper by combining the notion of fuzzy compact topological space (in Hutton's sense) and fuzzy topological ordered space, we introduce and study the concept of fuzzy compact ordered space.Its various properties are analyzed. We also develop and study order separation axioms called Ti separation axioms for fuzzy topological ordered spaces. The relationships between some of these Ti separation axioms with fuzzy compact ordered spaces are investigated.

1. Introduction

L. Nachbin in his famous book 'Topology and Order' published in 1965 [10] studied the relationship between topological and ordered structures. Mc Cartan [9] introduced T_i order separation axioms (i=1,2,3,4) in topological ordered spaces. To study of the interdependence between fuzzy topology and order, Katsaras [5] inroduced fuzzy topological ordered spaces in 1981.Here fuzzy topological ordered space is a triplet $(X, \mathcal{T}, \leq)$ where $\mathcal{T}$ is a fuzzy topology on X and $\leq$ is a crisp order on X.

Key Words : *Fuzzy topological ordered space, Fuzzy compactness, Separation axioms, Fuzzy Hausdorff space.*

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In this paper we provide with various separation axioms for fuzzy topological ordered spaces.For this ,we start with a definition of closed order that is different from Katsaras and develop fuzzy topological Hausdorff space.Further we develop fuzzy compact ordered spaces. Here we followed Chang's definition of fuzzy topology while ordering used on the space is crisp.

2. Preliminaries

Definition 2.1 : Let X be a nonempty set. A fuzzy topological ordered space is a triple $(X, \mathcal{T}, \leq)$ where $\mathcal{T}$ is a fuzzy topology on X and $\leq$ is a partial order on X.

Definition 2.2 : A fuzzy set μ in a fuzzy topological space $(X, \mathcal{T})$ is called a neighborhood of a point $x \in X$ if there exists an fuzzy open set μ_1,with $\mu_1 \leq \mu$ and $\mu_1(x) = \mu(x) > 0$.

A fuzzy set μ is open if and only if μ is a neighborhood of each $x \in X$ for which $\mu(x) > 0$.

Definition 2.3 : A function f from a fuzzy topological space $(X, \mathcal{T})$ to a fuzzy topological space $(Y, \mathcal{S})$ is called fuzzy continuous if $f^{-1}(\mu)$ is open in X for each open set μ in Y.

The function f is continuous at some $x \in X$ if $f^{-1}(\mu)$ is a neighborhood of x for each neighborhood μ of $f(x)$.

f is continuous on X iff f is continuous at each $x \in X$.

Definition 2.4 : A fuzzy set μ in a ordered set X is called

i) increasing if $x \leq y \Rightarrow \mu(x) \leq \mu(y)$

ii) decreasing if $x \leq y \Rightarrow \mu(x) \geq \mu(y)$

iii) order convex if $x \leq z \leq y \Rightarrow \mu(z) \geq (\mu(x) \wedge \mu(y))$

Note that : Constant functions are increasing,decreasing and order convex.

If μ is increasing then $1 - \mu$ is decreasing.

$\{\mu_i \mid i \in \Delta\}$ is increasing(resp.decreasing) then $\mu = \wedge\{\mu_i \mid i \in \Delta\}$is also increasing(resp. decreasing).

Definition 2.5 : Let μ be a fuzzy set in a ordered set X then the smallest increasing set containing μ, smallest decreasing set containing μ and smallest convex set containing μ

will be denoted by $i(\mu), d(\mu)$ and $c(\mu)$ respectively. Katsaras shown that
i) $i(\mu)(x) = \bigvee\{\mu(y) \mid y \leq x\}$.
ii) $d(\mu)(x) = \bigvee\{\mu(y) \mid y \geq x\}$.
iii) $c(\mu)(x) = \bigvee\{\mu(x_1) \wedge \mu(x_2) \mid x_1 \leq x \leq x_2\}$
Note that: $c(\mu) = i(\mu) \wedge d(\mu)$.
The smallest increasing closed set containing μ, the smallest decreasing closed set containing μ and the smallest convex closed set containing μ will be denoted by $I(\mu), D(\mu)$ and $C(\mu)$ respectively.
Definition 2.6 : Let λ be a fuzzy set of X and μ be a fuzzy set of Y then $\lambda \times \mu$ is fuzzy set of $X \times Y$,defined as

$$(\lambda \times \mu)(x, y) = \lambda(x) \wedge \mu(y) \text{ for each } (x, y) \in X \times Y$$

.

Definition 2.7 : Let X, Y be ordered sets and f be a function from X to Y.Then f is called increasing (resp. decreasing) if $x \leq y$ in X implies $f(x) \leq f(y)$(respectively $f(y) \leq f(x)$).
Proposition 2.1 : Let X, Y be ordered fuzzy topological spaces. A function $f : X \to Y$ is increasing if and only if $f^{-1}(\mu)$ is increasing in X for every increasing set μ in Y.
Proof : First, suppose that, $f : X \to Y$ is increasing and μ is increasing set in Y. Let $x \leq y$ in X. Then, $f^{-1}(\mu(x)) = \mu(f(x)), f^{-1}(\mu(y)) = \mu(f(y))$. Since f is increasing, $\mu(f(x)) \leq \mu(f(y))$. As μ is increasing, we have $\mu(f(x)) \leq \mu(f(y))$.
$\therefore f^{-1}(\mu)(x) \leq f^{-1}(\mu)(y)$.So,$f^{-1}(\mu)$ is an increasing function.
Similarly,we can prove the converse.

3. Fuzzy T_i Order Separation Axioms

Definition 3.1 : For $x \in X$ we define sets u_x and l_x as
$u_x = \{y \in X \mid x \leq y\}$ and $l_x = \{y \in X \mid y \leq x\}$
Definition 3.2 : A fuzzy topological ordered space $(X, \mathcal{T}, \leq)$ is called semiclosed ordered if for each $x \in X$, χ_{u_x} and χ_{l_x} are fuzzy closed sets.
Definition 3.3 : A fuzzy topological ordered space $(X, \mathcal{T}, \leq)$ is called fuzzy T_1 ordered iff the order $\leq$ on X is semiclosed.
Proposition 3.1 : A fuzzy topological ordered space $(X, \mathcal{T}, \leq)$ is fuzzy T_1 ordered iff

for $a, b \in X$ with $a \nleq b$,there exists increasing neighborhood λ of a such that $\lambda(a) > 0$ but $\lambda(b) = 0$ and there exists a decreasing neighborhood μ of b such that $\mu(b) > 0$ but $\mu(a) = 0$.

Proof : Suppose $(X, \mathcal{T}, \leq)$ is T_1 ordered. Then for $x \in X$, χ_{l_x} is fuzzy closed.
Let $a, b \in X$ with $a \nleq b$. By hypothesis,$\lambda = \chi_{X-l_b} \in \mathcal{T}$ and $\lambda(a) > 0, \lambda(b) = 0$.
Now let $x \leq y$. We want to show $\lambda(x) \leq \lambda(y)$.
If $\lambda(x) = 0$ then the result is obvious.
If $\lambda(x) > 0$ then $\lambda(x) = 1$. So,$x \in X - l_b$.$\therefore$ $x \nleq b$ which imply $y \nleq b$(because if $y \leq b$ then $x \leq y \Rightarrow x \leq b$, which is a contradiction)
Hence,$y \in X - l_b$ that is $\lambda(y) = 1$.
$\therefore \lambda(x) = \lambda(y)$.
So,λ is an increasing neighborhood of a such that $\lambda(a) > 0, \lambda(b) = 0$.
The other case may be treated similarly taking $\mu = \chi_{X-u_a}$.
Conversely,suppose for a pair $a \nleq b$ in X, there exists a decreasing $\mathcal{T}$-open neighborhood μ of b such that $\mu(a) = 0$. For each $b \in X - a$,we have $\chi_{X-u_a}(b) > 0$ So, $\mu \leq \chi_{X-u_a}$. Hence, χ_{X-u_a} is a neighborhood of b so that χ_{u_a} is closed. Similarly,χ_{l_b} is closed for each $a \in X$.
$\therefore (X, \mathcal{T}, \leq)$ is fuzzy T_1 ordered.

Corollary 3.1 : A fuzzy topological ordered space $(X, \mathcal{T}, \leq)$ is fuzzy T_1 ordered iff for $a, b \in X$with $a \neq b$, there exists a neighborhood λ of a such that $\lambda(a) > 0$ but $i(\lambda(b)) = 0$ and there exists a neighborhood μ of b such that $\mu(b) > 0$ but $d(\mu(a)) = 0$.

Proof : Follows from previous proposition.

Definition 3.4 : Let $(X, \leq)$ be a partially ordered set and $G = \{(x, y) \in X \times X \mid x \leq y\}$. Then G is called the graph of the partial order $\leq$.

Definition 3.5 : We say the order $\leq$ on X is closed if the characteristic function of G is a fuzzy closed set in in $(X \times X, \mathcal{T}')$ where $\mathcal{T}'$ is product fuzzy topology on $X \times X$.

Proposition 3.2 : An order $\leq$ on a fuzzy topological space is closed if and only if for any two points $x, y \in X$ with $x \nleq y$ there exists an increasing neighborhood λ of x and a decreasing neighborhood μ of y such that $\lambda \wedge \mu = 0$.

Proof : Suppose $(X, \mathcal{T}, \leq)$ is a fuzzy topological ordered space where the order $\leq$ is closed. Let $x \nleq y$ in X. Then,$(x, y) \notin G$ where G is the graph of the partial order $\leq$.
Since,χ_G is a fuzzy closed set in $(X \times X, \mathcal{T}'), 1 - \chi_G$ is fuzzy open set in $(X \times X, \mathcal{T}')$.

Now,$(x, y) \notin G$. So, $1 - \chi_G(x, y) = 1 > 0$.

$\therefore, 1 - \chi_G$ is a fuzzy open neighborhood of $(x, y) \in X \times X$.

Hence, we can find a fuzzy open set $\lambda \times \mu$ such that $\lambda \times \mu < (1 - \chi(G))$ where λ is a fuzzy open set such that $\lambda(x) > 0$ and μ is a fuzzy open set such that $\mu(y) > 0$.

Now we show $i(\lambda) \wedge d(\mu) = 0$.

For if there is $z \in X$ such that $(i(\lambda) \wedge d(\mu))(z) > 0$ then $i(\lambda)(z) \wedge d(\mu)(z) > 0$.

If $y \leq z \leq x$ then $z \leq x \Rightarrow i(\lambda)(x) > d(\mu)(z) > 0$. and$y \leq z \Rightarrow d(\mu)(y) > d(\mu)(z) > 0$

$\therefore i(\lambda)(x) > 0, d(\mu)(y) > 0$.Hence,$y \not\leq x$, which a contradiction.

Conversely,suppose the condition of the theorem is satisfied.To show χ_G is fuzzy closed set in $(X \times X, \mathcal{T}')$ Let $(x, y) \in 1 - \chi_G$ then $(1 - \chi_G)(x, y) > 0$ So, $\chi_G(x, y) < 1$ $\therefore$ $\chi_G(x, y) = 0$.

So, $(x, y) \notin G$ that is,$x \not\leq y$.

By hypothesis, there exists fuzzy open set $\lambda$and μ such that λ is increasing fuzzy open neighborhood of x and μ is a decreasing fuzzy open neighborhood of y and $\lambda \wedge \mu = 0$.

Clearly,$\lambda \times \mu$ is a fuzzy open neighborhood of (x, y) such that $\lambda \times \mu(x, y) > 0$.It is easy to verify that $\lambda \times \mu < 1 - \chi_G$. So $1 - \chi_G$ is a fuzzy open set.χ_G is closed. Hence,$\leq$ is a closed order.

Corollary 3.2 : The order $\leq$ defined on a fuzzy topological space X is closed if and only if for any two points $x, y \in X$ with $x \not\leq y$ there exists neighborhoods λ and μ of x and y respectively such that $i(\lambda) \wedge d(\mu) = 0$.

Proof : Follows from previous proposition.

Definition 3.6 : A fuzzy topological ordered space $(X, \mathcal{T}, \leq)$ is called a fuzzy T_2ordered space if the order $\leq$ is closed.

Definition 3.7 : A fuzzy topological space X is called a Hausdorff space if $x \neq y$ implies that there are neighborhoods μ, ρ of x, y respectively, with $\mu \wedge \rho = 0$.

Corollary 3.3 : Every fuzzy topological ordered space X with a closed order is a Hausdorff space.

Proof : Let $x, y \in X$ with $x \neq y$.Since the order of X is antisymmetric, at least one of $x \leq y$ and $y \leq x$ is false.Suppose that $x \leq y$ is false. Then, by the proposition x, y have disjoint neighborhoods as desired.

Proposition 3.3 : A fuzzy T_2 ordered space is fuzzy T_1 ordered.

Proof is straightforward.

Definition 3.8 : A fuzzy topological ordered space $(X, \mathcal{T}, \leq)$ is said to be fuzzy lower (resp. upper) regular ordered if for all closed decreasing fuzzy sets λ (resp. closed increasing) and for $x \in X$ with $\lambda(x) = 0$, there exists an increasing fuzzy open set μ (resp.decreasing fuzzy open) and decreasing open set ν (resp. increasing fuzzy open) such that $\mu(x) > 0, \lambda \leq \nu$ and $\mu \wedge \nu = 0$.
$(X, \mathcal{T}, \leq)$ is said to be fuzzy regular ordered iff it is both fuzzy lower and upper regular ordered.

$(X, \mathcal{T}, \leq)$ is said to be fuzzy lower (resp. upper) T_3 ordered iff it is fuzzy lower (resp. upper) T_1 ordered and fuzzy lower (resp. upper) regular ordered.
$(X, \mathcal{T}, \leq)$ is said to be fuzzy T_3 ordered iff it is fuzzy T_1 ordered and fuzzy regular ordered.

Proposition 3.4 : A fuzzy topological ordered space $(X, \mathcal{T}, \leq)$ is fuzzy lower (resp.upper) regularly ordered iff the following condition holds: For each $x \in X$ and an increasing (resp. decreasing)$\mathcal{T}$-open fuzzy neighborhood μ of x, there exists an increasing (resp.decreasing) $\mathcal{T}$ open set ν such that $\nu(x) > 0$ and $\nu \leq I(\nu) \leq \mu($ resp.$\nu \leq D(\nu) \leq \mu)$.

Proof : Suppose $(X, \mathcal{T}, \leq)$ is fuzzy lower (resp.upper) regular ordered space. Let $x \in X$ and let μ be an increasing (resp. decreasing)$\mathcal{T}$-open neighborhood of x, then $1 - \mu$ is $\mathcal{T}$-closed, decreasing (increasing)in X and $(1 - \mu)(x) = 0$.
By hypothesis there exists increasing (decreasing) fuzzy open set ν and decreasing (increasing) fuzzy open set λ such that $\nu(x) > 0, 1 - \mu \leq \lambda, \lambda \wedge \nu = 0$.Hence,$\nu \leq\ 1 - \lambda\ \leq \mu$ So,$I(\nu) \leq I(1 - \lambda) = 1 - \lambda$. Since $1 - \lambda$ is $\mathcal{T}$-closed,$\nu \leq I(\nu) \leq \mu(\nu \leq D(\mu) \leq \mu)$
Converse is straightforward.

Proposition 3.5 : If $(X, \mathcal{T}, \leq)$ is fuzzy lower or upper T_3 ordered then $(X, \mathcal{T}, \leq)$ is fuzzy T_2 ordered.

Proof : Let $x \not\leq y$ in X and suppose $(X, \mathcal{T}, \leq)$ is T_3-ordered then χ_{l_y} is $\mathcal{T}$-closed and decreasing and $\chi_{l_y}(x) = 0$. Since, $(X, \mathcal{T}, \leq)$ is regularly ordered ,there exists an increasing neighborhood μ of x and a decreasing neighborhood ν of l_y such that $\mu \wedge \nu = 0$.
Since,ν is a $\mathcal{T}$-neighborhood of l_y,it is a $\mathcal{T}$-neighborhood of y.
So,$(X, \mathcal{T}, \leq)$is fuzzy T_2 ordered.

Definition 3.9 : A fuzzy topological ordered space $(X, \mathcal{T}, \leq)$ is called a fuzzy normally ordered space iff the following condition is satisfied : Given a decreasing(resp. increasing) $\mathcal{T}$-closed fuzzy set μ and a decreasing (resp. increasing) $\mathcal{T}$-open fuzzy set ρ such that $\mu \leq \rho$,there exists a decreasing (resp.increasing) $\mathcal{T}$-open fuzzy set ρ_1 and a decreasing(resp. increasing)$\mathcal{T}$-closed fuzzy set μ_1 such that $\mu \leq \rho_1 \leq \mu_1 \leq \rho$.

Proposition 3.6 : $(X, \mathcal{T}, \leq)$ is a fuzzy normally ordered space iff the following condition is satisfied:

Given a decreasing (resp.increasing)closed fuzzy set μ and a decreasing (resp. increasing) $\mathcal{T}$-open fuzzy set ρ with $\mu \leq \rho$,there exists a decreasing(resp. increasing) open set ρ_1 such that $\mu \leq \rho_1 \leq D(\rho_1) \leq \rho$(resp. $\mu \leq \rho_1 \leq I(\rho_1) \leq \rho$).

Proof : Let $(X, \mathcal{T}, , \rho)$ be a fuzzy normally ordered space.let μ, ρ be given as in proposition.By definition,we have a decreasing fuzzy open set $\rho_1$and a decreasing fuzzy closed set μ_1 such that $\mu \leq \rho_1 \leq \mu_1 \leq \rho$.

Since,μ_1 is a decreasing fuzzy closed set such that $\rho_1 \leq \mu_1$ we have $\mu \leq \rho_1 \leq D(\rho_1) \leq \mu_1 \leq \rho$.

Conversely, suppose μ is a decreasing fuzzy closed set and ρ is a decreasing fuzzy open set such that $\mu \leq \rho$. Hence by condition of proposition,there exists a decreasing fuzzy open set ρ_1 such that $\mu \leq \rho_1 \leq D(\rho_1) \leq \rho$. Clearly, $D(\rho_1)$ is the smallest decreasing fuzzy closed set containing ρ_1. Put $\mu_1 = D(\rho_1)$. Then,$\mu \leq \rho_1 \leq \mu_1 \leq \rho$.

Proposition 3.7 : Every fuzzy normally ordered space is fuzzy regularly ordered space.

Proof : Suppose $(X, \mathcal{T}, \leq)$ be a normally ordered space.Let $x \in X$,μ be a decreasing $\mathcal{T}$-closed fuzzy set and ρ be a decreasing $\mathcal{T}$-open neighborhood of x with $\mu \leq \rho$. By normality, there exists a decreasing $\mathcal{T}$-open fuzzy set λ such that $\mu \leq \lambda \leq D(\lambda) \leq \rho$. So, $(x, \mathcal{T}, \leq)$ is fuzzy regularly ordered.

Definitiion 3.10 : A fuzzy normally ordered space which is also fuzzy T_1 ordered is called fuzzy T_4 ordered.

Corollary 3.4 : Every fuzzy T_4 ordered space is fuzzy T_3 ordered space.

Proof follows from proposition.

4. Fuzzy Compact Ordered Spaces

Compact Hausdorff spaces are important topological spaces because although these spaces are infinite, they admit approximations by finite sets.

Definition 4.1 : A fuzzy compact ordered space is a fuzzy compact topological space equipped with a closed order.

We know that,every fuzzy topological space with closed order is Hausdorff.This shows that a fuzzy compact ordered space is indeed a fuzzy compact Hausdorff space.

Proposition 4.1 : Let $(X, \mathcal{T}, \leq)$ be a fuzzy compact ordered space.If $K \subset X$ is a compact subset of X ,then $d(K)$ and $i(K)$ are fuzzy closed subsets of X.

Proof : proof Let $a \in X$ such that $d(K)(a) = 0$. Since $G(\leq)$ is closed $a \nleq x$ for every $x \in X$.

Then, we can find a increasing neighborhood λ_x of a and a decreasing neighborhood μ_x of x such that $\lambda_x \wedge \mu_x = 0$.

By the compactness of K there exists finitely many points $x_1, x_2, \ldots, x_n \in X$ such that $K \leq \mu_{x_1} \vee \mu_{x_2}, \vee \ldots, \vee \mu_{x_n}$

We define λ as $\lambda = \lambda_{x_1} \wedge \lambda_{x_2}, \wedge \ldots, \wedge \lambda_{x_n}$

Then λ is an increasing fuzzy neighborhood of point a. Further

$$\begin{aligned}
\lambda \wedge K &\leq \lambda \wedge (\mu_{x_1} \vee \mu_{x_2}, \vee \ldots, \vee \mu_{x_n}) \\
&= (\lambda \wedge \mu_{x_1}) \vee (\lambda \wedge \mu_{x_2}) \vee \ldots \vee (\lambda \wedge \mu_{x_n}) \\
&\leq (\lambda_{x_1} \wedge \mu_{x_1}) \vee (\lambda_{x_2} \wedge \mu_{x_2}) \vee \ldots \vee (\lambda_{x_n} \wedge \mu_{x_n}) \\
&= 0
\end{aligned}$$

Thus,$K \leq 1 - \lambda$. As, $1 - \lambda$ is decreasing , we get
$d(K) \leq 1 - \lambda$. Hence,$\lambda \wedge d(K) = 0$

Thus, $d(K)$ is closed.

Similarly, we can show that $i(K)$ is closed.

Proposition 4.2 : Let $(X, \mathcal{T}, \leq)$ be fuzzy compact ordered space,let λ be a decreasing (increasing) fuzzy subset of X and let μ be a fuzzy open set containing λ.Then,there exists a fuzzy open decreasing set ν such that $\lambda \leq \nu \leq \mu$.

Proof : We establish the result in the case that $\lambda$is decreasing.

Let $\nu = 1 - i(1 - \mu)$. By the preceding proposition $i(1 - \mu)$ is closed as $1 - \mu$ is closed subset of compact set, hence compact.

Since $i(1 - \mu)$ is increasing, ν is decreasing.

Thus,$i(1 - \mu) > 1 - \mu$ implies $\nu \leq \mu$.

Now, we show $\lambda \leq \nu$ that is $\lambda \wedge i(1 - \mu) = 0$.

Suppose there exists a point t such that $\lambda \wedge i(1-\mu)(t) > 0$

Then,$\lambda(t) > 0, i(1-\mu)(t) > 0$.

By the second condition there exists a point $x \in 1-\mu$ such that $t \geq x$

Since,$t \in \lambda, t \geq x$ we have $x \in \lambda$ as λ is decreasing.

This contradicts to $x \in 1-\mu$since μ is a neighborhood of λ.

Hence,$\lambda \leq \nu$.

Proposition 4.3 : Any closed subspace of a fuzzy compact ordered space is a fuzzy compact ordered space.

Proof : Let $(Y, \mathcal{T}_Y, \leq_Y)$ be a closed subspace of a fuzzy compact ordered space $(X, \mathcal{T}, \leq)$.

Let $u \leq \mathcal{T}_Y$ be an open cover of Y. Then, Y is a $\mathcal{T}$-closed implies that $1-Y$ is $\mathcal{T}$-open and $u = u'|_Y$, where$u \in u\sqcap' \in \mathcal{T}$.

$\therefore \{u' \mid u' \in \mathcal{T}\} \vee \{\mathit{1} - Y\}$ is an open cover of X.

Since,X is compact, it has a finite subcover.

$$\therefore X \leq u'_{\mathit{1}} \vee u'_{\mathit{2}} \vee \ldots \vee u'_n \vee \{\mathit{1} - Y\}$$

$$Y \leq u'_{\mathit{1}} \vee u'_{\mathit{2}} \vee \ldots \vee u'_n.$$

So, Y is compact fuzzy topological ordered space.

Definition 4.2 : Let $\{(X_t, \mathcal{T}_t, \leq_t) \mid t \in \Delta\}$ be a family of fuzzy topological ordered spaces. Let $X = \prod \{X_t \mid t \in \Delta\}$ and let $\mathcal{T}$ be the product fuzzy topology on X.Let $\leq \subset X \times X$ be defined as,for $x = (x_t)$ and $y = (y_t) \in X$,$x \leq y$ iff $x_t \leq_t y_t$ for all $t \in \Delta$. Then, $\leq$ is a partial order on X.

The fuzzy topological ordered space $(X, \mathcal{T}, \leq)$ is called the product fuzzy topological ordered space of the family $\{(X_t, \mathcal{T}_t, \leq_t) \mid t \in \Delta\}$.

Proposition 4.4 : Let $\{(X_t, \mathcal{T}_t, \leq_t) | t = 1, 2, \ldots, n\}$ be a finite family of fuzzy compact ordered spaces then the product fuzzy topological ordered space $(X, \mathcal{T}, \leq)$ is also a fuzzy compact ordered space.

Proof : Let $\{(X_t, \mathcal{T}_t, \leq_t) | t = 1, 2, \ldots, n\}$ be a finite family of compact fuzzy topological ordered spaces and $(X, \mathcal{T}, \leq)$ be the the product fuzzy topological ordered space.

By proposition the order $\leq$ is closed.

Let λ be a $\mathcal{T}$-closed fuzzy set with $\mathcal{U} = \{P_t^{-1}(U_t) | U_t \in \mathcal{T}_t, t = 1, 2, \ldots, n\}$ is an open cover of

$$\therefore \lambda \leq \sup\{(u) \mid u \in \mathcal{U}\}$$

Then,

$$P_t(\lambda) \leq \sup\{(V) \mid P_t^{-1}(V) \in \mathcal{U}\}$$

Choose,

$$\mathcal{U}_t = \{V_t \in \mathcal{T}_t \mid P_t^{-1}(V_t) \in \mathcal{U}\}$$

Then,

$$P_t(\lambda) \leq \sup\{V_t \mid V_t \in \mathcal{U}_t \text{and} P_t^{-1}(V_t) \in \mathcal{U}\}$$

$\therefore$ The fuzzy set $P_t(\lambda)$ is $\mathcal{T}_t$ closed.

Since,$\{(X_t, \mathcal{T}_t, \leq_t) | t = 1, 2, \ldots, n\}$ is compact fuzzy topological ordered space, each open cover $\mathcal{U}_t$ of $P_t(\lambda)$ has a finite subcover, say $\{u_1, u_2, \ldots, u_n\} \subset \mathcal{U}$.

Hence,$(X, \mathcal{T}, \leq)$ is compact fuzzy topological ordered space.

Proposition 4.5 : Every fuzzy compact ordered space $(X, \mathcal{T}, \leq)$ is fuzzy normally ordered space.

Proof : Let $(X, \mathcal{T}, \leq)$ be a fuzzy compact ordered space.Let λ be a closed fuzzy decreasing set and μ be open decreasing fuzzy set.

Let $\lambda \leq \mu$.

As $(X, \mathcal{T})$ is a normal topological space there exists a fuzzy open set δ and a fuzzy closed set σ such that $\lambda \leq \delta \leq \sigma \leq \mu$.

By previous proposition, there exists a decreasing open set ν such that $\lambda \leq \nu \leq \delta$

By proposition,$d(\sigma)$ is closed.

Thus,$\lambda \leq \nu \leq \delta \leq d(\sigma) \leq \mu$.

$\therefore \lambda \leq \nu \leq d(\sigma) \leq \mu$.

$\therefore (X, \mathcal{T}, \leq)$ is fuzzy normally ordered space.

Corollary 4.1 : Let $(X, \mathcal{T}, \leq)$ be a fuzzy compact topological ordered space then the set consisting of the open decreasing subsets and the open increasing subsets is a subbase for $\mathcal{T}$

Proof : Let $x, y \in X$ such that $x \nleq y$ then $d(\{x\}) \wedge i(\{y\}) = 0$.

As $(X, \mathcal{T}, \leq)$ is normally ordered, there exists a decreasing open fuzzy set λ containing $d(\{x\})$ and an increasing fuzzy open set μ containing $(i\{y\})$ such that $\lambda \wedge \mu = 0$.

Hence,the result, as X is Hausdorff.

Proposition 4.6 : Let $f$be a one-one,onto order preserving fuzzy continuous map from a fuzzy compact topological ordered space$(X, \mathcal{T}, \leq)$ to $(Y, \mathcal{S}, \leq')$. Then, $(Y, \mathcal{S}, \leq')$ is fuzzy compact topological ordered space.

Proof : As the order on X is closed and $f : X \to Y$ is order preserving fuzzy continuous function,the order $\leq'$ on Y is also closed.

Now it remains to show that $(X, \mathcal{T}, \leq)$ is compact.

Let $\mathcal{U} = \{u \mid u \in \mathcal{T}\}$ be an open cover of Y.

Then$f^{-1}(Y) \leq \sup\{f^{-1}(u) \mid u \in \mathcal{U}\}$

$\Rightarrow X \leq \sup\{f^{-1}(u)|(u) \in \mathcal{U}\}$

$\therefore$ $f^{-1}((u))$ is an open cover of X, but X is compact.so it has a finite subcover, say $\{f^{-1}((u_1)), f^{-1}((u_2)), \ldots, f^{-1}((u_n))\}$.

Since,f is onto ,$f(f^{-1}(u)) = u$

$$\begin{aligned} X &\leq f^{-1}(u_1) \vee f^{-1}(u_2) \vee \ldots \vee f^{-1}(u_n) \\ f(X) &\leq f(f^{-1}(u_1)) \vee f(f^{-1}(u_2)) \vee \ldots \vee f(f^{-1}(u_n)) \\ Y &\leq u_1 \vee u_2 \vee \ldots \vee u_n \end{aligned}$$

$\therefore$ the open cover $\mathcal{U}$ of Y has a finite subcover.

Hence, $(X, \mathcal{T}, \leq)$ is compact fuzzy topological ordered space.

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