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PRICING FORMUA FOR DERIVATIVE CONTRACT WITH DELAY

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Abstract

We consider pricing formula for derivative contracts whose dynamics is described by stochastic delayed differential equation (Sdde) with an exponential delay measure. For the purpose of finding pricing formula for the given derivative contract, we consider a term-structure model with a one-dimensional Brownian motion as the only source of randomness.
An explicit formula for the European call option is derived. We use martingale approach to derive such formulae.

1. Introduction

In this paper, we consider the problem of finding the fair(no-arbitrage) price, $V(t)$ for an interest rate derivative contract which pays the amount X at time t. The dynamic

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of the price of such contract is proposed to be a stochastic delay equation ($Sdde$). We discuss the modern approach to the pricing of an interest rate derivative using the theory of martingales to establish prices and hedging strategies. Such problem with different type delay model is discussed in [1]. Consider an interest rate derivative (stock) whose price at time t is given by stochastic process $S(t)$ whose dynamics is a stochastic delay differential equation:

$$dS(t) = \mu S(t-\delta)S(t)dt + (S(t) - \lambda Y(t))S(t)dW(t), \quad t \in [0,T],$$

$$S(t) = \varphi(t), \quad t \in [-\delta, 0]. \tag{1.1}$$

on the filtered probability space $(\Omega, F, F_t, P), t \in [0,T]$.
$\mu.\lambda, \delta$ are positive constants. W is a one-dimensional standard Brownian motion adapted to $(F_t)_{0\le t\le T}$

$$\varphi : \Omega \to C([-\delta, 0], \mathbb{R})$$

is F_0-measurable with respect to the Borel σ-field of the space $C([-\delta, 0], \mathbb{R})$.

$$Y(t) = \int_{-\delta}^{0} e^{\lambda S} S(t+s)ds.$$

Equation (1.1) has a pathwise unique solution S such that $S(t) > 0$ for all $t \ge 0$ whenever $\varphi(0) > 0$. (see [1], Thm.1).

2. Pricing Formula

Let the price of the option under consideration $S(t)$ satisfies the ($Sdde$) (1.1) and risk-free cash account, $B(t)$ satisfies the equation

$$dB(t) = r(t)B(t)dt, \quad B(0) = 1 \tag{2.1}$$

where r is the interest rate of return. Then the fair price, $V(t)$ at any time $t \in [0,T]$ is given by the formula

$$V(t) = E_Q[X|F_t^s]e^{-\int_t^T r(s)ds}. \tag{2.2}$$

We will discuss the proof of finding such fair price as well as the replicating portfolio. We start first by discussing the proof of the following theorem.

Theorem 2.1 : There exists a measure Q equivalent to P with

$$V(t) = E_Q[X|F_t^s]e^{-\int_t^T r(s)ds}.$$

Proof : We discuss the proof of Theorem 2.1 in the following steps:

Step (1) : Existence of an equivalent martingale measure:

We start as in [1]. Let $Z(t) = \frac{S(t)}{B(t)} = S(t)e^{-\int_0^t r(s)ds}, t \in [0,T]$ price process of our derivative contract. By the Ito's formula

$$\begin{aligned} dZ(t) &= \tfrac{1}{B(t)} dS(t) + S(t)(-r(t)B(t))dt \\ &= Z(t)[\{\mu S(t-\delta) - r(t)\}]dt + (S(t) - \lambda Y(t))dW(t). \end{aligned} \tag{2.3}$$

Let

$$\hat{S}(t) = \int_0^t \{\mu S(u-\delta) - r(u)\}du + \int_0^t (S(t) - \lambda Y(t))dW(t).$$

Then

$$dZ(t) = Z(t)d\hat{S}(t), \quad 0 < t < T \tag{2.4}$$

and

$$\hat{Z}(t) = \phi(0) + \int_0^t Z(u)d\hat{s}(u), \quad t \in [0,T]. \tag{2.5}$$

Define the process

$$\gamma(u) = \frac{(\mu S(u-\delta) - r(u)}{S(u) - \lambda Y(u)}, \quad u \in [0,T]$$

provided that $(S(u) - \lambda Y(u))$ is non-zero which implies that $r(u)$ for $0 \leq u \leq T$ is well defined.

By a backward conditioning argument as in [1], $E_P[u_T] = 1$, where

$$\begin{aligned} u_T \;=\; \exp&\left\{ -\int_0^T \frac{(\mu S(t-\delta) - r(u))}{(S(u) - \lambda Y(u))} dW(u) \right\} \\ &- \frac{1}{2} \int_0^T \left| \frac{(\mu S(t-\delta) - r(u))}{(S(u) - \lambda Y(u))} \right|^2 du \end{aligned}$$

(see [1], Section 3). By the Girsanov's theorem ([3], Thm. 5.5), the process

$$\tilde{w}(t) = w(t) + \int_0^t \gamma(u)du, \quad t \in [0,T]$$

is a standard wiener process under the probability measure Q and Q is defined by

$$dZ(t) = Z(t), \; (S(t) - \lambda Y(t))d\tilde{W} \quad t \in [0,T].$$

This implies that the discounted price process $Z(t)$ is a martingale under the equivalent measure Q hence the market under consideration $\{B(t), S(t)\}$ has no arbitrage property ([3], Thm. 7.1).

Step (2): Completeness of the market $\{B(t), S(t)\}, t \in [0,T]$ need one to prove that any contingent claim in this market is attainable to do so, let X be any contingent claim, i.e., an integrable non-negative random variable.
Define the process

$$M(t) = E_Q(B^{-1}(T)X|F_t^S) = E_Q(B^{-1}(T)X|F_t^{\tilde{W}}), \quad t \in [0,T].$$

Then $M(t), t \in [0,T]$, is an $(F_t^{\tilde{W}}) - Q$ martingale. By the martingale representation theorem ([3], Thm.9.4), there exists an $(F_t^{\tilde{W}})$ predictable process $g(t), t \in [0,T]$, such that

$$\int_0^T g(u)^2 du < \infty, \quad a.s.$$

and

$$M(t) = E_Q(B^{-1}(T)X) + \int_0^t g(u) d\tilde{W}(u), \quad t \in [0, mT].$$

Step (3) : Suppose that we employ the portfolio strategy which holds $\psi_1(t)$ units of the stock $S(t)$ and $\psi_2(t)$ units of cash account $B(t)$ at time t. The value of this portfolio for $0 \leq t \leq T$ is

$$V(t) := \psi_1(t)S(t) + \psi_2(t)B(t) = B(t)M(t)$$

and

$$dV(t) = B(t)dM(t) + M(t)dB(t) = \psi_2(t)dB(t) + \psi_1(t)dS(t), \quad t \in [0,T]$$

which means that the portfolio $(\psi_1(t), \psi_2(t_2))$ is a self-financing. The value of the portfolio at T, $V(T) = B(T)M(T) = X$. Hence the contingent claim X is attainable. Thus the market $\{B(t), S(t) : t \in [0,T]\}$ has no arbitrage if the price of the X is

$$V(t) = \frac{S(t)}{B(T)} E_Q(X|F_t^s)$$

at which $t \in [0,T]$ a.s. see [2].

3. Application of Theorem 2.1

As an application of Theorem 2.1, consider the derivative contract to be a European call option.

Theorem 2.2 : Let $V(t)$ be the fair price of a European call option with exercise price K and maturity time T. Then for all $t \in [T$
$del, T]$, the fair price $V(t)$ is

$$V(t) = S(t)\Phi(B_+(t)) - K\Phi(B_-(t))e^{-\int_t^T r(s)ds}$$

where $\Phi(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{u^2}{2}} du, y \in \mathbb{R}$ is the distribution function of the standard normal law:

$$B_{\pm}(t) = \log \frac{S(t)}{K} + \frac{\int_t^T (r(u) \pm \frac{1}{2}(S(u) - \lambda Y(u))^2 du}{(\int_t^T (S(u) - \lambda Y(u))^2 du)^{\frac{1}{2}}}.$$

If $T > \delta$ and $t < T - \delta$, then

$$V(t) \quad = \quad e^{\int_0^t r(s)ds} E_Q(H(\tilde{S}(T-\delta), -\frac{1}{2}\int_{T-\delta}^{T} (S(u) - \lambda Y(u))^2 du \int_{T-\delta}^{T} (S(u) - \lambda Y(u))^2 du | F_t),$$

where H is given by

$$H(x, m, \sigma^2) := xe^{m + \frac{\sigma^2}{2}} \Phi(\alpha_1, (x, m, \sigma)) - K\Phi(\alpha_2(x, m, \sigma))e^{-\int_0^T r(s)ds}$$

and

$$\alpha_1(x, m, \sigma) := \frac{1}{\sigma}\left[\log\left(\frac{x}{K}\right) + \int_0^T r(s)ds + m + \sigma^2\right]$$

$$\alpha_2(x, m, \sigma) := \frac{1}{\sigma}\left[\log\left(\frac{x}{K}\right) + \int_0^T r(s)ds + m\right]$$

here $\sigma, x \in \mathbb{R}^+, m \in \mathbb{R}$. The replicating portfolio is

$$\psi_1(t) = \Phi(B_+(t), \Psi_2(t) = -K\Phi(B_-(t)e^{-\int_t^T r(s)ds}, \quad t \in [T - \delta, T].$$

Remark : Since in one factor models for the term structure of interest rates in continuous-time framework the free rate $r(t)$ assumed to be a time homogeneous Markov process

$$dr(t) = a(r(t))dt + b(r(t))dW(t),$$

and the volatility of the given stock is delayed by the model for $r(t)$, it is aproperiate for the delayed model we treated in this paper is to assume that the free rate $r(t)$ is to be modeled by the following $(Sdde)$

$$dr(t) = \mu(x_t - \lambda Y_t)^2 dt + \theta(x_t - \lambda Y_t)dW(t), \quad \text{for} \quad t \geq 0 \quad r_0 = r$$

where

$$Y_t = \int_{-\delta}^{0} e^{\lambda s} ds x_t = x_t^0, \quad t \leq 0$$

for some $\theta > 0, \lambda > 0$ and $x \in \mathbb{R}$. $x_t = \varphi$ for $t \in [-\delta, 0]$-deterministic bounded measurable function. Since $x_t - \lambda Y_t$ is the deviation of the logarithm of the present value of the process x from its exponentially weighted average λY, then the model for $r(t)$ we suggest will fit with the model for delayed stock model.

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References

[1] Arriojas M., Hu Y., Mohammed S. E. A. and Pup G., A delayed Black and Scholes formula 1, Priprint, pp. 10.

[2] Baxter M. and Rennie A., Financial Calculus, Cambridge University Press (1996).

[3] Kallianpur G. and Karandikar R. J., Introduction to Option Pricing Theory, Birkhaüser, Boston-Basel-Berlin (2000).

[4] Mohammed S.E.A., Stochastic differential systems with memory: Theory, examples and applications. Progress in probability 42, Birkhaüser (1998), 1-77.

[5] Andrew J. G. Cairns, Interest Rate Models, introduction, Princeton University Press (2004).